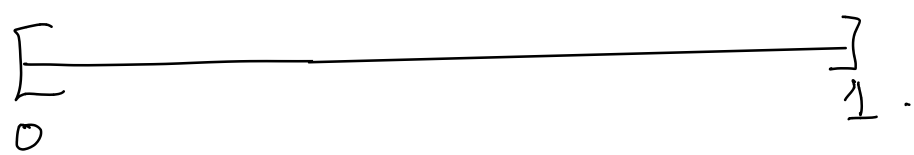
Lecture #14.

The Poisson Random Variable.

Suppose we have the unit interval



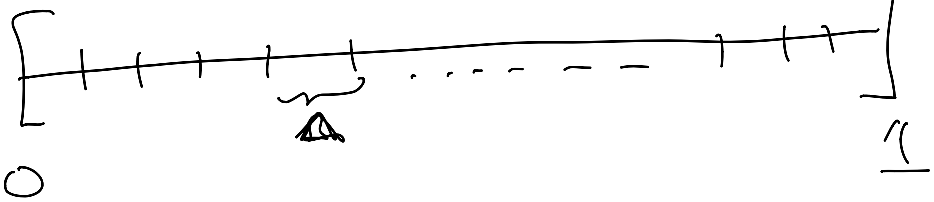
and we know that some event <u>can</u> occur randomly throughout the interval. Suppose, on average, the number of events appearing in the interval is λ. Let X = # of events in $[0,1]$.

Such a RV is Poisson.

How do we compute the pmf?

Consider dividing the interval into n pieces, each of length $1/n$.

Call each subinterval Δ.



- For large enough n, the probability that there is more than one event in a given Δ is ≈ 0.
- Therefore, each Δ is a Bernoulli trial with probability $\frac{\lambda}{n}$ of having an event.

Thus, $X \approx \underbrace{Bin(n, \frac{\lambda}{n})}$.

pmf $\Rightarrow$ $p_n(x) = \binom{n}{i}\left(\frac{\lambda}{n}\right)^i\left(1-\frac{\lambda}{n}\right)^{n-i}$

So letting $n \to \infty$ we get

$$\boxed{P(X=i) = p(x) = e^{-\lambda}\frac{\lambda^i}{i!}}$$

Any DRV with $P(X=i) = e^{-\lambda}\frac{\lambda^i}{i!}$ is called <u>Poisson</u>.

By the same argument (but in reverse) we may approximate the binomial random variable. That is, for n large and $\lambda = np$ "moderately large" (ie for p small)

if $X = Bin(n,p)$, then

$$P(X=i) \approx e^{-\lambda} \frac{\lambda^i}{i!}$$

Ex: A real estate firm sells, on average, 2 houses per day. What is the probability that they sell at least 16 houses next week?

Soln: If the firm sells 2 houses per day on average, then the average per week (7 days) is $2 \cdot 7 = 14$ houses per week.

Thus, if we let

X = # of houses sold per week,

then X is a Poisson RV with $\lambda = 14$.

Therefore, our pmf is

$$p(i) = P(X=i) = e^{-14}\left(\frac{14^i}{i!}\right).$$

and we want $P(X \geq 16) = \underbrace{\sum_{i=16}^{\infty} e^{-14}\left(\frac{14^i}{i!}\right)}_{\text{might be hard.}}$

So compute

$$P(X \geq 16) = 1 - P(X \leq 15)$$

$$= 1 - \sum_{i=0}^{15} e^{-14}\left(\frac{14^i}{i!}\right)$$

$$= 0.3306 \approx 33\%$$

Suppose that X is Poisson. λ:

What is $E[X]$?

$$E[X] = \sum_{i=0}^{\infty} i \frac{e^{-\lambda}\lambda^i}{i!}$$

$$= \sum_{i=1}^{\infty} \frac{e^{-\lambda}\lambda^i}{(i-1)!}$$

$$= \lambda e^{-\lambda} \sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!} = \lambda e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^j}{j!}$$

$$= \lambda e^{-\lambda} e^{\lambda} = \underline{\lambda}$$

which we may have guessed from the first analysis.

Another way to see this, if X is poisson, then X is the limit of $Bin(n, p)$ where $p = \frac{\lambda}{n}$. Therefore,

$$E[X] = \lim_{n\to\infty} E[Bin(n, \tfrac{\lambda}{n})]$$
$$= \lim_{n\to\infty} \cancel{n}\left(\frac{\lambda}{\cancel{n}}\right) = \lambda.$$

By a similar analysis,

$$Var(X) = \lim_{n\to\infty} Var\left(Bin\left(n, \frac{\lambda}{n}\right)\right)$$
$$= \lim_{n\to\infty} \cancel{n}\left(\frac{\lambda}{\cancel{n}}\right)\left(1 - \frac{\lambda}{n}\right)$$
$$= \lambda \lim_{n\to\infty}\left(1 - \frac{\lambda}{n}\right) = \lambda.$$

(Though you can just derive $Var(X)$ from definition directly. see text).

So for X Poisson, $E[X] = Var(X) = \lambda$.

<u>Ex:</u> A company makes electric motors. Each motor has a probability of 0.01 of being defective. What is the probability that the next 300 of the production line will contain 5 defective motors?

<u>Soln</u>: This is really a binomial problem!
Each motor has a probability of being defective 0.1,
so the RV $X = \#$ of defective in 300 is
binomial, hence

$$P(X=5) = \binom{300}{5}(0.01)^5 (0.99)^{295} \approx 0.10099.$$

<u>However</u>, since $n = 300 >> 0.1 = p$, we can approx.
with poisson! If each motor has probability 0.01
of being defective, then the average number of defects
motors in a run of 300 is $\lambda = 300 \times 0.01 = \underline{\underline{3}}$

Thus, $$P(X=i) \approx e^{-3}\left(\frac{3^i}{i!}\right).$$

and so $$P(X=5) = e^{-3}\left(\frac{3^5}{5!}\right) = 0.10082.$$

(so error of $\sim 0.001\%$.).